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LETTER TO THE EDITOR

Mixed population Minority Game with generalized strategies

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Abstract. We present a quantitative theory, based on crowd effects, for the market volatility in a Minority Game played by a mixed population. Below a critical concentration of generalized strategy players, we find that the volatility in the crowded regime remains above the random coin-toss value regardless of the 'temperature' controlling strategy use. Our theory yields good agreement with numerical simulations.

Challet and Zhang's Minority Game (MG) offers a simple paradigm in the study of complex adaptive systems such as financial markets [1–8]. In the MG an odd number N of agents, each with s strategies and a memory of size m, repeatedly compete to be in the minority. The basic MG features agents who use their highest scoring strategy. As pointed out by Marsili *et al* [9]†, a probabilistic strategy choice reflects a particular behavioural model and has a long tradition in economics. Cavagna *et al* [8] performed numerical simulations of the MG in which agents use an exponential probability weighting controlled by a 'temperature' T; this is called the Thermal Minority Game (TMG) although it has been noted that T^{-1} may instead correspond to the agents' learning rate (see [11]). Challet *et al*, in addition to presenting a detailed spin-glass theory for the basic MG [2], have recently identified problems [12] with the TMG results of Cavagna *et al* [8]. Our own interest in the TMG has focused on the finding that the volatility (i.e. standard deviation) σ can be reduced from being larger than the random coin-toss value ('worse-than-random') to being smaller than the random coin-toss value ('better-than-random') just by altering the relative probability weighting [8]. We recently provided an analytic theory which explains this effect in terms of crowds [13].

In this Letter, we consider a generalized Minority Game in which a concentration q of agents employ such probabilistic strategy selection at each turn of the game. We present a quantitative theory, based on crowd effects, which yields good agreement with numerical simulations. We find that below a critical concentration q_c^*, the volatility σ remains larger than the random coin-toss value regardless of the 'temperature' T controlling the strategy selection.

Our generalized Minority Game contains N agents who choose repeatedly between option 0 (e.g. buy) and option 1 (e.g. sell). The winners are those in the minority group, e.g. sellers win if there is an excess of buyers. The outcome at each timestep represents the winning decision, 0 or 1. A common bit-string of the m most recent outcomes [14]‡ is made available

† T-dependent, Boltzmann-like strategy weightings were discussed by Marsili at the International Workshop on Econophysics and Statistical Finance (Palermo, September 1998) as reported in [10].

‡ See Challet and Marsili [14] and references therein for demonstrations confirming the relevance of the actual memory in the MG, in contrast to the claim of Cavagna [15].

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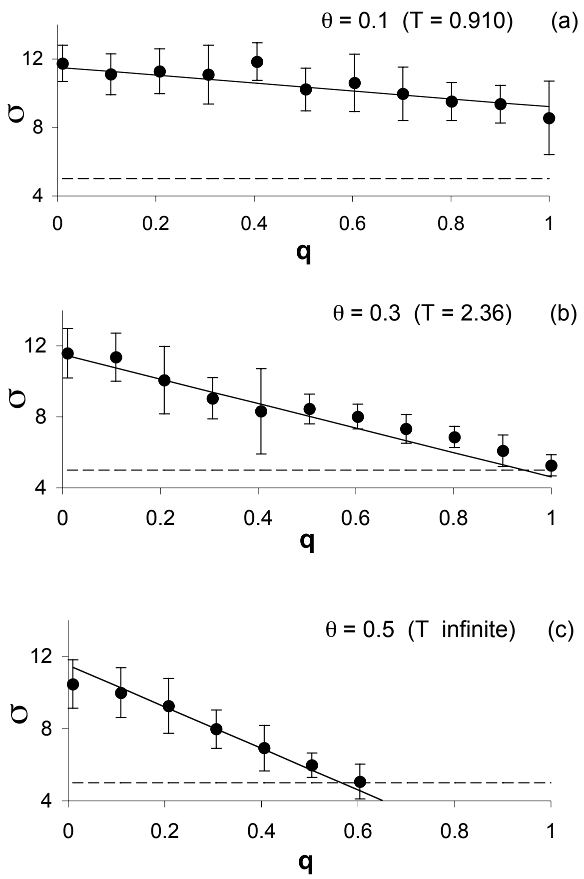

Figure 1. Comparison between numerical simulations (solid circles with error bars obtained over many runs) and the present theory (solid line using equation (6)) for the volatility σ as a function of TMG agent concentration q at fixed θ: (*a*) $\theta = 0.1$; (*b*) $\theta = 0.3$; (*c*) $\theta = 0.5$. The 'temperature' T corresponding to each θ is given. $N = 101$ and $m = 2$. The dashed line shows the random coin-toss value.

to the agents at each timestep. The agents randomly pick s strategies at the beginning of the game, with repetitions allowed, from the pool of all possible strategies. We focus on $s = 2$. After each turn, the agent assigns one (virtual) point to each of his strategies which would have predicted the correct outcome. In the basic MG, each agent plays the most successful strategy in his possession, i.e. the one with the most virtual points. Here we instead allow a concentration q of agents to follow a more general behavioural model: in particular, these agents play their worst strategy with probability θ, and hence play their best strategy with probability $(1 - \theta)$. These qN agents will be called 'TMG agents' because of the direct connection with the Thermal Minority Game [8]†. The remaining $(1 - q)N$ agents choose their best strategy with probability unity (i.e. $\theta = 0$ as in the basic MG); hence they will be called 'MG agents'.

† The Thermal Minority Game discussed in [8] depends on a parameter T (or equivalently $1/\beta$) called a 'temperature'. We could similarly define T by setting the probability of playing the worst strategy $\theta = \mathrm{e}^{-\beta}/(\mathrm{e}^{\beta} + \mathrm{e}^{-\beta})$. Hence $T = 2[\ln(\theta^{-1} - 1)]^{-1}$. $T = 0$ corresponds to $\theta = 0$ while $T \to \infty$ corresponds to $\theta \to 1/2$, and hence we will only consider $0 \leqslant \theta \leqslant 1/2$ in this Letter.

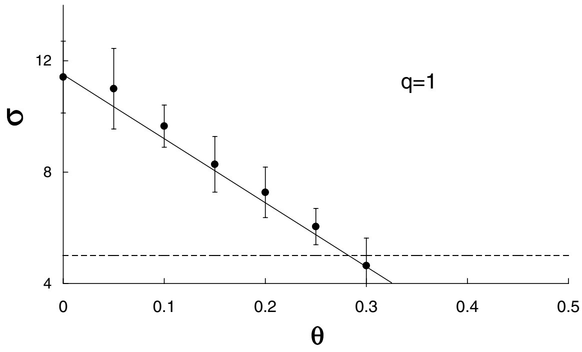

Figure 2. Comparison between numerical simulations (solid circles with error bars obtained over many runs) and the present theory (solid line using equation (6)) for the volatility σ as a function of the probability θ for a pure population of TMG agents (i.e. $q = 1$). $N = 101$ and $m = 2$. the dashed line shows the random coin-toss value.

Figure 1 shows the volatility σ obtained from numerical simulations of a game with $N = 101$ and $m = 2$, as a function of q at various fixed θ values. The dashed line shows the random coin-toss value for N agents, given by $\sqrt{N}/2$. Figure 2 shows an example of the corresponding numerical results for σ as a function of θ at fixed q. A definite trend can be seen in figures 1 and 2, despite the numerical spread which arises naturally for different runs: as the concentration q of TMG agents increases, or the probability θ (i.e. T) increases, the volatility σ decreases. At $q = 1$ (figure 2) we reproduce the main finding of [8] whereby σ falls from worse-than-random to better-than-random with increasing θ ('temperature' T). The numerical results in figure 1 indicate that below a critical q, σ lies in the worse-than-random regime regardless of T. Our goal is to develop a quantitative theory describing the trend in the run-averaged volatility (i.e. the volatility averaged over initial strategy configurations) as a function of q and θ.

In [6] we presented a quantitative theory for the volatility σ in the basic MG which yields good agreement with numerical simulations over the entire parameter range of interest. The theory is based on the consideration of the combined actions of crowds and their anticorrelated partners (anticrowds). For each crowd–anticrowd pair, the action of the anticrowd will effectively nullify the action of the crowd if they are of similar size, hence reducing the volatility σ [5, 6]. For small m and large N [6], the crowds are typically much larger than the anticrowds [6] hence the basic MG is in the 'crowded' regime (i.e. σ is larger than the random coin-toss value); this is the regime of interest here since we are focusing on the transition of σ from worse-than-random to better-than-random. Although the present numerical results correspond to $N = 101$, any N is suitable such that the system remains in this 'crowded' regime [6]. A cruder version of our crowd theory was earlier shown to provide a good quantitative description for the MG played by a population of mixed-memory agents [7, 14]. Given this success, we build the present theory using the same crowd–anticrowd ideas. Consider any two strategies r and r^* within the list of 2^{m+1} strategies in the reduced strategy space [1, 6]. At any moment in the game, the strategies can be ranked according to their virtual points, $r = 1, 2 \ldots 2^{m+1}$ where $r = 1$ is the best strategy, $r = 2$ is second best, etc. Note that in the small-m regime of interest, the strategy ranking in order of decreasing virtual points can be taken to be identical to the strategy ranking in order of decreasing number of users (i.e. decreasing popularity) to a good approximation [6]. Accidental degeneracies may arise whereby two different strategies momentarily have identical virtual points; however, these degeneracies are removed when considering an average over several timesteps—hence any agent holding two strategies with the same ranking must necessarily have picked the same strategy twice. Let $p(r, r^*|r^* \geqslant r)$ be the probability that a given agent picks r and r^*, where $r^* \geqslant r$. Let $p(r, r^*|r^* \leqslant r)$ be the probability that a given agent picks r and r^*, where $r^* \leqslant r$. The probability that a TMG agent plays r is given by

$$\begin{aligned} p_r^{\mathrm{TMG}} &= \sum_{r^*=1}^{2^{m+1}} [\, \theta\ p(r, r^*|r^* \leqslant r) + \ (1-\theta)\ p(r, r^*|r^* \geqslant r)] \\ &= \theta\ p_-(r) + 2^{-2(m+1)}\ \theta + \ (1-\theta)\ p_+(r) \end{aligned} \tag{1}$$

where $p_+(r) = \sum_{r^*} p(r, r^*|r^* \geqslant r)$ is the probability that the agent has picked r *and* that r is the agent's best (or equal best) strategy; $p_-(r) = \sum_{r^*} p(r, r^*|r^* < r)$ is the probability that the agent has picked r *and* that r is the agent's worst strategy. The factor $2^{-2(m+1)}$ in equation (1) originates from $p(r, r^*|r^* = r)$. The probability that an MG agent plays r is given by

$$p_r^{\mathrm{MG}} = p_+(r). \tag{2}$$

It follows that $p_+(r) + p_-(r) = p(r)$ where

$$p(r) = 2^{-m}(1 - 2^{-(m+2)}) \tag{3}$$

is the probability that an agent holds strategy r after his $s = 2$ picks with no condition on whether it is best or worst. Now we consider the mean number of agents n_r playing strategy r in the mixed-population game containing a concentration q of TMG agents and $(1-q)$ of MG agents. This is given by

$$\begin{aligned} n_r &= \ q\ N\ p_r^{\mathrm{TMG}} + \ (1-q)\ N\ p_r^{\mathrm{MG}} \\ &= \ N\ (1 - 2\ q\ \theta)\ p_+(r) + N\ q\ \theta\ p(r) + 2^{-2(m+1)}\ N\ q\ \theta. \end{aligned} \tag{4}$$

If n_r agents all use strategy r, they will act as a 'crowd', i.e. they make the same decision. If $n_{\bar{r}}$ agents simultaneously use the strategy $\bar{r}$ anticorrelated to r, they will make the opposite (anticorrelated) decision and hence act as an 'anticrowd' [6]. The standard deviation $\sigma(q, \theta)$ in the number of agents making a particular decision (say 0) is given by [6]

$$\sigma(q, \theta) = \left[\frac{1}{2} \sum_{r=1}^{2^{m+1}} \frac{1}{4} |n_r - n_{\bar{r}}|^2 \right]^{\frac{1}{2}}. \tag{5}$$

Using equations (3), (4) and (5) for r and $\bar{r} = 2^{m+1} + 1 - r$, we obtain

$$\sigma(q, \theta) = [1 - 2\ q\ \theta]\ \{\sigma(q, \theta)\}_{q\theta=0} \tag{6}$$

where $\{\sigma(q, \theta)\}_{q\theta=0}$ is just the standard deviation for the basic MG (i.e. $q = 0$ and/or $\theta = 0$). In [6], we provided an analytic formulation of $\{\sigma(q, \theta)\}_{q\theta=0}$. However, equation (6) is more general in that it does not specify the level of approximation used to obtain $\{\sigma(q, \theta)\}_{q\theta=0}$.

Our theory (equation (6)) predicts that the effect on the volatility caused by a change in population composition and/or 'temperature' can be described by a simple prefactor $[1-2q\theta]$. Provided that the basic MG is in the crowded regime as discussed earlier, equation (6) should hold for all N and m and hence any value of $\{\sigma(q, \theta)\}_{q\theta=0}$. Hence we can predict the critical value q_{c} for fixed θ, or θ_{c} for fixed q, at which $\sigma(q, \theta)$ crosses from worse-than-random to better-than-random. For a given value of θ, it follows from equation (6) that

$$q_c(\theta) = \frac{1}{2\theta} - \frac{\sqrt{N}}{4\theta} \frac{1}{\{\sigma(q, \theta)\}_{q\theta=0}}. \tag{7}$$

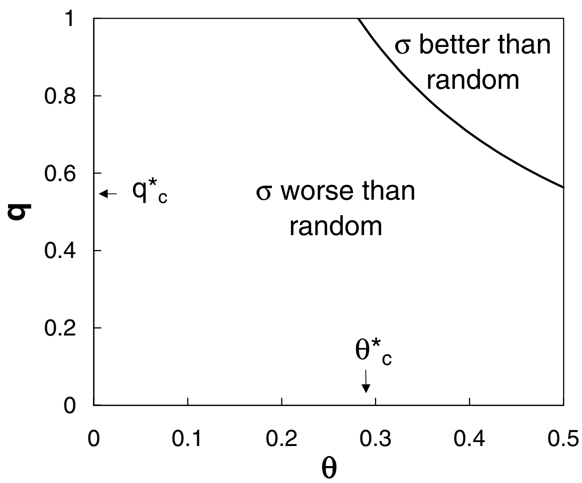

Figure 3. 'Phase diagram' in (q, θ) space. The curve corresponds to equation (7) and separates regions where volatility σ lies above the random coin-toss value (worse-than-random) and below (better-than-random). $N = 101$ and $m = 2$.

A similar expression follows for $\theta_c(q)$. Given that $0 \leqslant \theta \leqslant 1/2$, equation (7) implies that the run-averaged numerical volatility should lie above the random coin-toss value if $q < q_c^*$ where

$$q_c^* = 1 - \frac{\sqrt{N}}{2} \frac{1}{\{\sigma(q, \theta)\}_{q\theta=0}} \tag{8}$$

regardless of 'temperature' T. Since we are considering N and m values such that the basic MG is in the worse-than-random regime, $\{\sigma(q, \theta)\}_{q\theta=0} \geqslant \sqrt{N}/2$ and therefore $0 \leqslant q_c^* \leqslant 1$ as required. Similarly $\sigma(q, \theta)$ will remain above the random coin-toss value for all q if $\theta < \theta_c^*$ where

$$\theta_c^* = \frac{1}{2} - \frac{\sqrt{N}}{4} \frac{1}{\{\sigma(q, \theta)\}_{q\theta=0}}. \tag{9}$$

Figures 1 and 2 compare the present theory (equation (6)) to the numerical simulations. The theoretical values lie within the numerical spread for a wide range of q and θ values, and hence provide a quantitative explanation of the observed trends. Since we are interested in testing the simple prefactor scaling predicted by equation (6), we have generated figures 1 and 2 using the numerical value of $\{\sigma(q, \theta)\}_{q\theta=0}$ obtained from the basic MG; we emphasize, however, that an analytic formulation for $\{\sigma(q, \theta)\}_{q\theta=0}$ is provided in [6]. Although not relevant for the main results of this paper, the present theory (equation (6)) begins to underestimate the numerical results in the better-than-random regime as $\sigma(q, \theta) \to 0$ (not shown). There are shortcomings in the theory which can explain this effect; in particular, p_r^{TMG} in equation (1) is an average value over the configuration space of possible initial strategy picks, and over time. It has a decreasing dependence on r as $\theta \to 0.5$, hence giving rise to $\sigma = 0$ (i.e. exact crowd–anticrowd cancellation) for $q = 1$ and $\theta = 0.5$. Consider $q = 1$ and $\theta = 0.5$; for a particular configuration of strategies picked at the start of the game, and at a particular moment in time, the number of agents using each strategy is typically distributed *around* the value $N\ 2^{-(m+1)}$. It is this non-flat distribution describing the strategy-use by coin-flipping TMG agents which will actually give rise to a non-zero σ. Having obtained σ for a given initial configuration of strategies, the average should then be taken over all initial strategy configurations. In [13] we provide a fuller discussion of the behaviour of both the numerical and theoretical results in the better-than-random regime, and present an analytic calculation which accounts for the saturation of the numerical values below the random coin-toss limit. Figure 3 shows the theoretical 'phase diagram' for the volatility $\sigma(q, \theta)$. The curve $q_c(\theta)$, or equivalently $\theta_c(q)$, separates the regions where σ is worse-than-random and better-than-random. Also indicated are q_c^* and θ_c^*.

In summary, we have analysed a mixed population Minority Game with generalized strategies. The main feature of the numerical results regarding volatility reduction from worse-than-random to better-than-random can be explained quantitatively without having to solve the detailed game dynamics. More generally, it is clear that there will be some properties of MG games which cannot be described using such time- and configuration-averaged theories as used here (see [11]). Moreover, the volatility in real financial markets is more likely to correspond to a *single* run which evolves from a specific initial configuration of agents' strategies. Our crowd–anticrowd viewpoint can, however, be extended to deal with these game dynamics via the dynamical equations governing the co-evolution of the crowd–anticrowd populations. The correct equations are not continuous in time in general. The MG dynamics described in terms of the time evolution of crowds–anticrowds are presented elsewhere [16].

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